

On the Quantum Origin of the Mixmaster Chaos Covariance

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Abstract

Our analysis shows how the covariant chaotic behavior characterizing the evolution of the Mixmaster cosmology near the initial singularity can be taken as the semiclassical limit in the canonical quantization performed by the corresponding Hamiltonian representation.

1 Cosmological Framework

Though the Standard Cosmological Model (SCM) is based on the highly symmetric Friedmann-Lemaître-Robertson-Walker (FLRW) solution of the Einstein equations, nevertheless don't exist neither theoretically neither experimentally any evidences to prevent that our universe underwent a more general dynamics in the very early stages of its evolution, and only in a later phase it isotropized reaching a complete agreement with actual experimental data.

The simplest generalization of the FLRW dynamics consists of the so-called Bianchi models, whose anisotropic evolution, for some of them, can be represented as a FLRW model plus a gravitational waves packet [1, 2]. Among this classification, the types VIII¹ and IX appear as the most general ones allowed by the homogeneity constraint² and their asymptotic evolution toward the cosmological singularity manifests a chaotic-like behavior [3]. The cosmological interest lies in the IX model (the so-called Mixmaster [4]), which has the same space symmetries as the closed FLRW universe, and whose dynamics allows the line element to be decomposed as

$$ds^2 = ds_0^2 - \delta_{(a)(b)} G_{ik}^{(a)(b)} dx^i dx^k \quad (1)$$

where ds_0 denotes the line element of an isotropic universe having constant positive curvature, $G_{ik}^{(a)(b)}$ is a set of spatial tensors³ and $\delta_{(a)(b)}(t)$ are amplitude functions, resulting small sufficiently

¹All the considerations we will develop for the type IX apply also to the VIII one since, close to the singularity, they have the same morphology.

²IX's geometry is invariant under the $SO(3)$ group.

³These tensors satisfy the equations

$$G_{ik}^{(a)(b);l}{}_{;l} = -(n^2 - 3)G_{ik}^{(a)(b)}, \quad G_{;k}^{(a)(b)(b)k} = 0, \quad G_i^{(a)(b)(b)i} = 0,$$

far from the singularity.

We dedicate our analysis to find a precise relation between the Mixmaster deterministic chaos and the quantum behavior characterizing the Planckian era, showing how the invariant measure for the former, provided general Misner-Chitré-like coordinates (MCl) [5, 6], is independent of the time gauge [7] and coincides with the stationary probability for the semiclassical limit of the latter.

2 Billiard Representation

Using generic MCl variables (ξ, θ, τ) [7], the dynamics is described by the two-dimensional canonical variational principle

$$\delta \int (p_\xi \xi' + p_\theta \theta' - f' \mathcal{H}_{ADM}) d\eta = 0, \quad (2)$$

where f is a generic function,

$$\mathcal{H}_{ADM} = \sqrt{\varepsilon^2 + U}, \quad \varepsilon^2 = q^2 p_\xi^2 + \frac{p_\theta^2}{q^2} \quad (3)$$

where $U(\xi, \theta, \eta)$ denotes the corresponding potential term and $q = q(\xi) \equiv \sqrt{\xi^2 - 1}$. Moreover the equation for the temporal gauge reads

$$N(\eta) = \frac{12D}{\mathcal{H}_{ADM}} e^{2f} \frac{df}{d\tau} \tau' \quad (4)$$

so that our analysis remains fully independent of the choice of the time variable until the form of f and τ' is not fixed.

For the following developments it is of key interest the following relation

$$\frac{d(\mathcal{H}_{ADM} f')}{d\eta} = \frac{\partial(\mathcal{H}_{ADM} f')}{\partial \eta}. \quad (5)$$

The function $f(\eta)$ plays the role of a parametric function of time and actually the anisotropy parameters [5][7] H_i ($i = 1, 2, 3$) are functions of the variables (ξ, θ) only [7]. In the domain Γ_H where all the H_i are simultaneously greater than 0, the potential term U can be modeled by the potential walls⁴

$$U_\infty = \Theta_\infty(H_1) + \Theta_\infty(H_2) + \Theta_\infty(H_3) \quad (6)$$

therefore, by (5), in Γ_H the ADM Hamiltonian becomes (asymptotically) an integral of motion

$$\forall \{\xi, \theta\} \in \Gamma_H \left\{ \begin{array}{l} \frac{\partial \mathcal{H}_{ADM}}{\partial f} = \frac{\partial E}{\partial f} = 0 \\ \mathcal{H}_{ADM} = \sqrt{\varepsilon^2 + U} \cong \varepsilon = E \end{array} \right. \quad (7)$$

where E is a constant. In view of (7) the variational principle (2) reduces to

$$\delta \int (p_\xi d\xi + p_\theta d\theta - E df) = \delta \int (p_\xi d\xi + p_\theta d\theta) = 0. \quad (8)$$

in which the Laplacian is referred to the geometry of the sphere of unit radius.

⁴ $\Theta_\infty(x) = \begin{cases} +\infty & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$

Following the standard Jacobi procedure to reduce this principle to a geodesic one we get, for the closed domain region Γ_H , the Riemannian line element

$$ds^2 = E^2 \left[\frac{d\xi^2}{\xi^2 - 1} + (\xi^2 - 1) d\theta^2 \right]. \quad (9)$$

Since the above metric (9) has curvature scalar $R = -\frac{2}{E^2}$ the point-universe moves over a negatively curved bidimensional space on which the potential wall (6) cuts the region Γ_H ; indeed the invariant Lyapunov exponent for the dynamical flux associated to (9) reads $\lambda_v = 1/E > 0$ [7]. The point-universe, bouncing on the potential walls, is reflected from a geodesic to another one, making each of them unstable. By itself, the positivity of Lyapunov number is not enough to ensure the system chaoticity, since its derivation remains valid for any Bianchi type model, but for the Mixmaster case the potential walls reduce the configuration space to a compact region (Γ_H), ensuring a real chaotic behavior.

3 Statistical Mechanics Approach

For a Statistical Mechanics reformulation of the dynamics, we adopt in (2) the restricted time gauge $\tau' = 1$, leading to the variational principle

$$\delta \int \left(p_\xi \frac{d\xi}{df} + p_\theta \frac{d\theta}{df} - \mathcal{H}_{ADM} \right) df = 0. \quad (10)$$

In spite of this restriction, for any assigned time variable τ (i.e. η) there exists a corresponding function $f(\tau)$ (i.e. a set of MCI variables able to provide the scheme presented in Section 2) defined by the (invertible) relation

$$\frac{df}{d\tau} = \frac{\mathcal{H}_{ADM}}{12D} N(\tau) e^{-2f}. \quad (11)$$

Hence the analysis to derive the invariant measure for the system follows the same lines presented in [5, 6]. Indeed we got a suitable representation of the Mixmaster chaoticity in terms of a two-dimensional point-universe randomizing within Γ_H , admitting an “energy-like” constant of motion $\varepsilon = E$, then well-described by a *microcanonical ensemble*, whose Liouville invariant measure reads⁵

$$d\rho = A \delta(E - \varepsilon) d\xi d\theta dp_\xi dp_\theta, \quad A = const.. \quad (12)$$

After the natural positions

$$p_\xi = \frac{\varepsilon}{q} \cos \phi, \quad p_\theta = \varepsilon q \sin \phi, \quad (13)$$

being $0 \leq \phi < 2\pi$, and the integration over all values of ε ⁶, we arrive to the uniform invariant measure [10, 5][8]

$$d\mu = w_\infty(\xi, \theta, \phi) d\xi d\theta d\phi \equiv \frac{1}{8\pi^2} d\xi d\theta d\phi. \quad (14)$$

The key point of our analysis is that any stationary solution of the Liouville theorem, like (14), remains valid for any choice of the time variable τ ; indeed in [6] the construction of the

⁵ $\delta(x)$ denotes the Dirac function.

⁶The dependence on the initial conditions doesn't contain any information about the system chaoticity.

Liouville theorem with respect to the variables (ξ, θ, ϕ) shows the existence of such properties even for the invariant measure (14).

More precisely, in agreement with the analysis presented in [6], during a free geodesic motion the asymptotic functions $\xi(f)$, $\theta(f)$ and $\phi(f)$ are provided by the simple system

$$\frac{d\xi}{df} = q \cos \phi, \quad \frac{d\theta}{df} = \frac{\sin \phi}{q}, \quad \frac{d\phi}{df} = -\frac{\xi \sin \phi}{q} \quad (15)$$

and therefore over the reduced phase space⁷ $\{\xi, \theta\} \otimes S_\phi^1$ the distribution w_∞ behaves like the step-function

$$w_\infty(\xi, \theta, \phi) = \begin{cases} \frac{1}{8\pi^2} & \forall \quad \{\xi, \theta, \phi\} \in \Gamma_H \otimes S_\phi^1 \\ 0 & \forall \quad \{\xi, \theta, \phi\} \notin \Gamma_H \otimes S_\phi^1 \end{cases} \quad (16)$$

stationary solution of the Liouville theorem

$$q \cos \phi \frac{\partial w_\infty}{\partial \xi} + \frac{\sin \phi}{q} \frac{\partial w_\infty}{\partial \theta} - \frac{\xi \sin \phi}{q} \frac{\partial w_\infty}{\partial \phi} = 0. \quad (17)$$

If now we restrict our attention to the distribution function on the configuration space Γ_H

$$\varrho(\xi, \theta) \equiv \int_0^{2\pi} w_\infty(\xi, \theta, \phi) d\phi, \quad (18)$$

by (17) we get for such reduced form the two dimensional continuity equation

$$q \cos \phi \frac{\partial \varrho_\infty}{\partial \xi} + \frac{\sin \phi}{q} \frac{\partial \varrho_\infty}{\partial \theta} = 0 \quad (19)$$

and the *microcanonical* solution on the whole configuration space $\{\xi, \theta\}$ then reads

$$\varrho_\infty(\xi, \theta) = \begin{cases} \frac{1}{4\pi} & \forall \quad \{\xi, \theta\} \in \Gamma_H \\ 0 & \forall \quad \{\xi, \theta\} \notin \Gamma_H \end{cases}. \quad (20)$$

4 Quantum Origin of the Chaos

The main result of the above Section 3 [7, 8] is the proof that the chaoticity of the Bianchi IX model above outlined is an intrinsic feature of its dynamics and not an effect induced by a particular class of references. Appearing this intrinsic chaos close to the Big Bang, we infer that it has strict relations with the indeterministic quantum dynamics the model performs in the *Planckian era*. The link between quantum and deterministic chaos is searched in the sense of a semiclassical limit for the canonical quantization of the model.

Indeed the asymptotical principle (2) can be quantized by a natural Schrödinger approach [9]

$$i\hbar \frac{\partial \psi}{\partial \tau} = \hat{\mathcal{H}}_{ADM} \psi, \quad (21)$$

⁷ S_ϕ^1 denotes the ϕ -circle.

being $\psi = \psi(\tau, \xi, \theta)$ the wave function for the point-universe and, implementing $\hat{\mathcal{H}}_{ADM}$ (see (3)) to an operator⁸, i.e.

$$\begin{aligned}\xi &\rightarrow \hat{\xi}, & \theta &\rightarrow \hat{\theta}, \\ p_\xi &\rightarrow \hat{p}_\xi \equiv -i\hbar \frac{\partial}{\partial \xi}, & p_\theta &\rightarrow \hat{p}_\theta \equiv -i\hbar \frac{\partial}{\partial \theta},\end{aligned}\quad (22)$$

the equation (21) rewrites explicitly, in the asymptotic limit $U \rightarrow U_\infty$,

$$i \frac{\partial \psi}{\partial \tau} = \sqrt{\hat{\varepsilon}^2 + \frac{U_\infty}{\hbar^2}} \psi, \quad (23)$$

where we left U_∞ to stress that the potential cannot be neglected on the entire configuration space $\{\xi, \theta\}$ and, being infinity out of Γ_H , it requires as boundary condition for ψ to vanish outside the potential walls $\psi(\partial\Gamma_H) = 0$. Since the potential walls U_∞ are time independent, a solution of (23) can be taken in the form

$$\psi(\tau, \xi, \theta) = \sum_{n=1}^{\infty} c_n e^{-iE_n \tau / \hbar} \varphi_n(\xi, \theta) \quad (24)$$

where c_n are constant coefficients and we assumed a discrete “energy” spectrum because the quantum point-universe is restricted in the finite region Γ_H . Once taken an appropriate symmetric normal ordering prescription, the squared relation from (23) and the position (24) lead to the eigenvalue problem

$$\begin{aligned}\left[-q \frac{\partial}{\partial \xi} q \frac{\partial}{\partial \xi} - \frac{1}{q} \frac{\partial}{\partial \theta} \frac{1}{q} \frac{\partial}{\partial \theta} \right] \varphi_n &= \\ &= \frac{E_n^2 - U_\infty}{\hbar^2} \varphi_n \equiv \frac{E_{\infty n}^2}{\hbar^2} \varphi_n.\end{aligned}\quad (25)$$

The *quantum* equation (23) is equivalent to the Wheeler-DeWitt one for the same Bianchi model, once separated the positive and negative frequency solutions, with the advantage that now τ is a real time variable.

In what follows we search the semiclassical solution of this equation regarding the eigenvalue $E_{\infty n}$ as a finite constant (i.e. we consider the potential walls as finite) and only at the end of the procedure we will take the limit for U_∞ (6).

We infer that in the semiclassical limit when $\hbar \rightarrow 0$ and the *occupation number* n tends to infinity (but $n\hbar$ approaches a finite value) the wave function $\varphi_n \rightarrow \varphi$ and $E_{\infty n} \rightarrow E_\infty$ so that we have

$$\varphi(\xi, \theta) = \sqrt{r(\xi, \theta)} \exp \left\{ i \frac{S(\xi, \theta)}{\hbar} \right\}, \quad (26)$$

where r and S are functions to be determined.

Substituting (26) in (25) and separating the real from the complex part we get two independent

⁸The non vanishing canonical commutation relations are

$$[\hat{\xi}, \hat{p}_\xi] = i\hbar, \quad [\hat{\theta}, \hat{p}_\theta] = i\hbar.$$

equations, i.e.

$$E_\infty^2 = \underbrace{q^2 \left(\frac{\partial S}{\partial \xi} \right)^2 + \frac{1}{q^2} \left(\frac{\partial S}{\partial \theta} \right)^2}_{\text{classical term}} + \underbrace{\dots}_{O(\hbar^2)} \quad (27)$$

$$0 = \underbrace{q \frac{\partial}{\partial \xi} \left(q r \frac{\partial S}{\partial \xi} \right) + \frac{1}{q^2} \frac{\partial}{\partial \theta} \left(r \frac{\partial S}{\partial \theta} \right)}_{O(1/\hbar)} . \quad (28)$$

The dominant term in (27) reduces to the Hamilton-Jacobi equation and its solution can be easily checked to be⁹

$$S(\xi, \theta) = \int \left\{ \frac{1}{q} \sqrt{E_\infty^2 - \frac{k^2}{q^2}} d\xi + k d\theta \right\} \quad (29)$$

where k is an integration constant. Through the identifications

$$\frac{\partial S}{\partial \xi} = p_\xi, \quad \frac{\partial S}{\partial \theta} = p_\theta \iff S = \int (p_\xi d\xi + p_\theta d\theta) , \quad (30)$$

(27) is reduced to a mere algebraic constraint which is the asymptotic one $\mathcal{H}_{ADM}^2 = E^2 \equiv E_\infty^2$ and is solved by (13) replacing $\varepsilon = E_\infty$, whose compatibility with (29) and (30) is then obtained using the equations of motion (15) which provide

$$\frac{d\xi}{d\phi} = -\frac{\xi^2}{\xi^2 - 1} \text{ctg} \varphi \Rightarrow \sqrt{\xi^2 - 1} \sin \varphi = c , \quad (31)$$

where $c = \text{const.}$; the required compatibility comes from the identification $k = E_\infty c$. Since $E_\infty \rightarrow i\infty$ outside Γ_H the solution $\varphi(\xi, \theta)$ vanishes out of the billiard.

The substitution in (28) of the positions (13) leads to the new equation

$$q \cos \phi \frac{\partial r}{\partial \xi} + \frac{\sin \phi}{q} \frac{\partial r}{\partial \theta} = 0 , \quad (32)$$

which coincides with (19), provided the identification $r \equiv \varrho_\infty$; this result ensures the correspondence between the statistical and the semiclassical quantum analysis.

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⁹The discontinuity of this function on the boundary of Γ_H is due to the model adopted and doesn't affect the probability distribution.

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